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Positivity of Dunkl's Intertwining Operator via the Trigonometric Setting

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1 Introduction

In [15], it was proven that Dunkl's intertwining operator between the rational Dunkl operators for a fixed finite reflection group and nonnegative multiplicity function is positive. As a consequence, we obtained an abstract Harish-Chandra-type integral representation for the Dunkl kernel, the image of the usual exponential kernel under the intertwiner. The proof was based on methods from the theory of operator semigroups and a rank-one reduction.

In the present paper, we give a new, completely different proof of these results under the only additional assumption that the underlying reflection group has to be crystallographic. In contrast to the proof of [15], where precise information on the supports of the representing measures could only be obtained by going back to estimates of the kernel from [5], this information is now directly obtained. Our new approach relies first on an asymptotic relationship between the Opdam-Cherednik kernel and the Dunkl kernel as recently observed by de Jeu [6], and second on positivity results of Sahi [17] for the Heckman-Opdam polynomials and their nonsymmetric counterparts.

2 Preliminaries

2.1 Basic notation

Let $\mathfrak{a}$ be a finite-dimensional Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$. We use the same notation for the bilinear extension of $\langle \cdot, \cdot \rangle$ to the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$, and we

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identify $\mathfrak{a}$ with its dual $\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ via the given inner product. For $\alpha \in \mathfrak{a}\setminus\{0\}$, we write $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ and $\sigma_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha$ for the orthogonal reflection in the hyperplane perpendicular to α. We consider a crystallographic root system R in $\mathfrak{a}$, that is, R is a finite subset of $\mathfrak{a}\setminus\{0\}$ which spans $\mathfrak{a}$ and satisfies $\sigma_\alpha(R) = R$ and $\langle\alpha, \beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$. We also assume that R is indecomposable and reduced, that is, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$. Let W be the finite reflection group generated by the σ_α, $\alpha \in R$. We will fix a positive subsystem R_+ of R as well as a *nonnegative* multiplicity function $k = (k_\alpha)_{\alpha\in R}$, satisfying $k_\alpha = k_\beta$ if α and β are in the same W-orbit.

2.2 Rational Dunkl operators and Dunkl's intertwiner

References for this section are [7, 8, 12, 15]. Let $\mathcal{P} = \mathbb{C}[\mathfrak{a}]$ denote the vector space of complex polynomial functions on $\mathfrak{a}$, and $\mathcal{P}_n \subset P$ the subspace of polynomials which are homogeneous of degree $n \in \mathbb{Z}_+$. The rational Dunkl operators on $\mathfrak{a}$ associated with R and fixed multiplicity $k \geq 0$ are given by

$$T_\xi = T_\xi(k) = \partial_\xi + \sum_{\alpha\in R_+} k_\alpha \langle\alpha, \xi\rangle \frac{1}{\langle\alpha, \cdot\rangle}(1 - \sigma_\alpha), \quad \xi \in \mathfrak{a}. \tag{2.1}$$

These operators commute and map $\mathcal{P}$ onto itself. Moreover, there exists a unique linear isomorphism $V = V_k$ on $\mathcal{P}$ with $V(1) = 1$, $V(\mathcal{P}_n) = P_n$, and $T_\xi V = V\partial_\xi$ for all $\xi \in \mathfrak{a}$. According to [8], the intertwining operator V can be extended to larger classes of functions as follows: for $r > 0$, let $K_r := \{x \in \mathfrak{a} : |x| \leq r\}$ denote the ball of radius r and define

$$A_r := \left\{ f : K_r \longrightarrow \mathbb{C},\ f = \sum_{n=0}^{\infty} f_n \text{ with } f_n \in \mathcal{P}_n, \|f\|_{A_r} := \sum_{n=0}^{\infty} \|f_n\|_{\infty,K_r} < \infty \right\}, \tag{2.2}$$

where $\|f_n\|_{\infty,K_r} := \sup_{x\in K_r} |f_n(x)|$. The space A_r is a Banach space with norm $\|\cdot\|_{A_r}$ (in fact, a commutative Banach algebra). V extends to a continuous linear operator on A_r by $V(\sum_{n=0}^\infty f_n) := \sum_{n=0}^\infty Vf_n$. The Dunkl kernel Exp_W is defined by

$$\mathrm{Exp}_W(\cdot, z) := V\big(e^{\langle\cdot, z\rangle}\big), \quad z \in \mathfrak{a}_\mathbb{C}. \tag{2.3}$$

It extends to a holomorphic function on $\mathfrak{a}_\mathbb{C} \times \mathfrak{a}_\mathbb{C}$ which is symmetric in its arguments. For $\lambda \in \mathfrak{a}_\mathbb{C}$, $\mathrm{Exp}_W(\lambda, \cdot)$ is the unique holomorphic solution of the joint eigenvalue problem

$$T_\xi f = \langle\lambda, \xi\rangle f \quad \forall \xi \in \mathfrak{a},\ f(0) = 1. \tag{2.4}$$

For $x \in \mathfrak{a}$, we denote by $C(x)$ the closure of the convex hull of the W-orbit Wx of x in $\mathfrak{a}$. Moreover, for a locally compact Hausdorff space X, we write $M^1(X)$ for the set of probability measures on the Borel σ-algebra of X. In [15], the following is proven.

Theorem 2.1. For each $x \in \mathfrak{a}$, there exists a unique probability measure $\mu_x \in M^1(\mathfrak{a})$ such that

$$Vf(x) = \int_{\mathfrak{a}} f(\xi) d\mu_x(\xi) \quad \forall f \in A_{|x|}. \tag{2.5}$$

The support of μ_x is contained in $C(x)$. □

As a consequence,

$$\mathrm{Exp}_W(x, z) = \int_{\mathfrak{a}} e^{\langle \xi, z \rangle} d\mu_x(\xi) \quad \forall z \in \mathfrak{a}_{\mathbb{C}}. \tag{2.6}$$

In [15], the proof of the inclusion $\operatorname{supp} \mu_x \subseteq C(x)$ requires the exponential bounds on Exp_W from [5], which are by far not straightforward. As is well known, the W-invariant parts of the rational and trigonometric Dunkl theories are, for certain discrete sets of multiplicities, realized within the classical Harish-Chandra theory for semisimple symmetric spaces. In particular, for such k, the generalized Bessel functions

$$J_W(\cdot, z) = \frac{1}{|W|} \sum_{w \in W} \mathrm{Exp}_W\left(\cdot, w^{-1} z\right), \quad z \in \mathfrak{a}_{\mathbb{C}}, \tag{2.7}$$

can be identified with the spherical functions of an underlying Cartan motion group; for details see, for example, [6]. In this case, their integral representation according to (2.6) is a special case of a (Euclidean type) Harish-Chandra integral, and the inclusion $\operatorname{supp} \mu_x \subseteq C(x)$ follows from Kostant's convexity theorem [11, Proposition IV.4.8 and Theorem IV.10.2].

2.3 Cherednik operators and the Opdam-Cherednik kernel

The basic concepts of this as well as the following section are due to Opdam [13] (see also [14, part I]), Heckman (see [10, part I]), and Cherednik [4]. Let

$$P := \left\{ \lambda \in \mathfrak{a} : \left\langle \lambda, \alpha^\vee \right\rangle \in \mathbb{Z} \; \forall \alpha \in R \right\} \tag{2.8}$$

denote the weight lattice associated with the root system R. For $\lambda \in \mathfrak{a}_{\mathbb{C}}$, we define the exponential e^λ on $\mathfrak{a}_{\mathbb{C}}$ by $e^\lambda(z) := e^{\langle \lambda, z \rangle}$ and denote by $\mathcal{T}$ the $\mathbb{C}$-span of $\{e^\lambda, \; \lambda \in P\}$. This is the algebra of trigonometric polynomials on $\mathfrak{a}_{\mathbb{C}}$ with respect to R. The Cherednik operator in direction $\xi \in \mathfrak{a}$ is defined by

$$D_\xi = D_\xi(k) = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-\alpha}} (1 - \sigma_\alpha) - \langle \rho(k), \xi \rangle, \tag{2.9}$$

where $\rho(k) = (1/2)\sum_{\alpha\in R_+} k_\alpha \alpha$. Each D_ξ maps $\mathcal{T}$ onto itself and (for fixed k) the operators D_ξ commute. Notice that in contrast to the rational T_ξ, they depend on the particular choice of R_+. For each $\lambda \in \mathfrak{a}_\mathbb{C}$, there exists a unique holomorphic function $G(\lambda, \cdot)$ in a tubular neighborhood of $\mathfrak{a}$ which satisfies

$$D_\xi G(\lambda, \cdot) = \langle \lambda, \xi\rangle G(\lambda, \cdot) \quad \forall \xi \in \mathfrak{a}, \ G(\lambda, 0) = 1 \tag{2.10}$$

(see [14, Corollary I.7.6]). G is called the Opdam-Cherednik kernel. It is in fact (as a function of both arguments) holomorphic in a suitable tubular neighborhood of $\mathfrak{a}_\mathbb{C} \times \mathfrak{a}$ [13, Theorem 3.15]. The rational Dunkl operators can be considered a scaling limit of the Cherednik operators, and this implies limit relations between the kernels Exp_W and G. We will need the following variant of [6, Theorem 4.12].

Proposition 2.2. Let $\delta > 0$ be a constant, $K, L \subset \mathfrak{a}_\mathbb{C}$ compact sets, and $h : (0, \delta) \times L \to \mathfrak{a}_\mathbb{C}$ a continuous mapping such that $\lim_{\epsilon\to 0} \epsilon h(\epsilon, \lambda) = \lambda$ uniformly on L. Then

$$\lim_{\epsilon\to 0} G\big(h(\epsilon, \lambda), \epsilon z\big) = \mathrm{Exp}_W(\lambda, z) \tag{2.11}$$

uniformly for $(\lambda, z) \in L \times K$. □

The proof is the same as for [6, Theorem 4.12], with λ/ϵ replaced by $h(\epsilon, \lambda)$. We mention that for integral k, such a limit transition has first been carried out in [2] by use of shift operator methods.

2.4 A scaling limit for nonsymmetric Heckman-Opdam polynomials

The definition of these polynomials involves a suitable partial order on P; we refer to the one used in [14]. Let

$$P_+ := \left\{\lambda \in P : \left\langle \lambda, \alpha^\vee \right\rangle \ge 0 \, \forall \alpha \in R_+\right\} \tag{2.12}$$

denote the set of dominant weights associated with R_+, and λ_+ the unique dominant weight in the orbit $W\lambda$. One defines $\lambda \lhd \nu$ if either $\lambda_+ < \nu_+$ in dominance ordering (i.e., $\nu_+ - \lambda_+ \in Q_+$, the $\mathbb{Z}_+$-span of R_+), or if $\lambda_+ = \nu_+$ and $\nu < \lambda$ (in dominance ordering). Further, $\lambda \unlhd \nu$ means $\lambda = \nu$ or $\lambda \lhd \nu$. The nonsymmetric Heckman-Opdam polynomials $\{E_\lambda : \lambda \in P\} \subset \mathcal{T}$ associated with R_+ and k are uniquely characterized by the conditions

$$E_\lambda = \sum_{\nu \unlhd \lambda} a_{\lambda,\nu} e^\nu \quad \text{with } a_{\lambda,\lambda} = 1, \tag{2.13}$$

$$D_\xi E_\lambda = \langle \widetilde{\lambda}, \xi\rangle E_\lambda \quad \forall \xi \in \mathfrak{a}, \tag{2.14}$$

with the shifted spectral variable $\widetilde{\lambda} = \lambda + (1/2)\sum_{\alpha\in R_+} k_\alpha \epsilon(\langle\lambda, \alpha^\vee\rangle)\alpha$. Here $\epsilon : \mathbb{R} \to \{\pm 1\}$ is defined by $\epsilon(x) = 1$ for $x > 0$ and $\epsilon(x) = -1$ for $x \leq 0$. For details, see [13] and [14, Section I.2.3].

On the other hand, we know (cf. (2.10)) that $G(\widetilde{\lambda}, \cdot)$ is the up-to-a-constant-factor unique holomorphic solution of (2.14). Hence

$$E_\lambda = c_\lambda \cdot G(\widetilde{\lambda}, \cdot), \tag{2.15}$$

with a constant $c_\lambda = E_\lambda(0) > 0$. The precise value of c_λ is given in [14, Theorem 4.7].

Corollary 2.3. For $\lambda \in P$ and $z \in \mathfrak{a}_\mathbb{C}$,

$$\mathrm{Exp}_W(\lambda, z) = \lim_{n\to\infty} \frac{1}{c_{n\lambda}} E_{n\lambda}\left(\frac{z}{n}\right). \tag{2.16}$$

The convergence is locally uniform with respect to z. □

Proof. Fix $\lambda \in P$ and observe that $\widetilde{n\lambda} = n\lambda + (1/2)\sum_{\alpha\in R_+} k_\alpha \epsilon(\langle\lambda, \alpha^\vee\rangle)\alpha$ for all $n \in \mathbb{N}$. Thus by Proposition 2.2 and identity (2.15), we have, locally uniformly for $z \in \mathfrak{a}_\mathbb{C}$,

$$\mathrm{Exp}_W(\lambda, z) = \lim_{n\to\infty} G\left(\widetilde{n\lambda}, \frac{z}{n}\right) = \lim_{n\to\infty} \frac{1}{c_{n\lambda}} E_{n\lambda}\left(\frac{z}{n}\right). \tag{2.17}$$

■

Remark 2.4. Similar results hold for the symmetric Heckman-Opdam polynomials

$$P_\lambda(z) = \frac{|W\lambda|}{|W|} \sum_{w\in W} E_\lambda(w^{-1}z), \quad \lambda \in P_+. \tag{2.18}$$

They are W-invariant and related with the multivariable hypergeometric function

$$F(\lambda, z) = \frac{1}{|W|} \sum_{w\in W} G(\lambda, w^{-1}z) \tag{2.19}$$

via

$$P_\lambda = c_\lambda^* \cdot F(\lambda + \rho, \cdot) \quad \forall \lambda \in P_+ \tag{2.20}$$

with $c_\lambda^* = |W\lambda| \cdot c_\lambda$ and $\rho = \rho(k) = (1/2)\sum_{\alpha\in R_+} k_\alpha \alpha$, (cf. [10, equation (4.4.10)]). This also follows from (2.15) because F is in fact W-invariant in both arguments and for $\lambda \in P_+$, the shifted weight $\widetilde{\lambda}$ is contained in the W-orbit of $\lambda + \rho$ [13, Proposition 2.10]. Further, Corollary 2.3 implies that for $\lambda \in P_+$ and $z \in \mathfrak{a}_\mathbb{C}$,

$$J_W(\lambda, z) = \lim_{n\to\infty} F\left(n\lambda + \rho, \frac{z}{n}\right) = \lim_{n\to\infty} \frac{1}{c_{n\lambda}^*} P_{n\lambda}\left(\frac{z}{n}\right). \tag{2.21}$$

For illustration, consider the rank-one case (type A_1) with $\mathfrak{a} = \mathbb{R}$ and $R_+ = \{2\alpha\}$, $\alpha = 1$. Fix $k = k_{2\alpha} \geq 0$. Then according to the example in [13, page 89f],

$$\begin{aligned} F(\lambda, z) &= {}_2F_1\left(a, b, c; \frac{1}{2}(1-\cosh z)\right), \\ G(\lambda, z) &= {}_2F_1\left(a, b, c; \frac{1}{2}(1-\cosh z)\right) \\ &\quad + \frac{a}{2c}\sinh z \cdot {}_2F_1\left(a+1, b+1, c+1; \frac{1}{2}(1-\cosh z)\right), \end{aligned} \tag{2.22}$$

with $a = \lambda + k$, $b = -\lambda + k$, and $c = k + 1/2$. The weight lattice is $P = \mathbb{Z}$, and the associated Heckman-Opdam polynomials are given by

$$\begin{aligned} P_n(z) &= c_n^* F(n+k, z) = c_n^* \cdot Q_n^k(\cosh z), \quad n = 0, 1, \ldots, \\ E_n(z) &= c_n G(\widetilde{n}, z) = c_n \left[Q_{|n|}^k(\cosh z) + \frac{\widetilde{n}+k}{2k+1} \cdot \sinh z Q_{|n|-1}^{k+1}(\cosh z)\right], \quad n \in \mathbb{Z}, \end{aligned} \tag{2.23}$$

with $\widetilde{n} = n + k$ for $n > 0$, $\widetilde{n} = n - k$ for $n \leq 0$, and the renormalized Gegenbauer polynomials

$$Q_n^k(x) = {}_2F_1\left(n+2k, -n, k+\frac{1}{2}; \frac{1}{2}(1-x)\right). \tag{2.24}$$

Relation (2.21) reduces to the classical limit

$$\lim_{n\to\infty} Q_n^k\left(\cos\frac{z}{n}\right) = j_{k-1/2}(z) \quad (z \in \mathbb{C}) \tag{2.25}$$

for the modified Bessel functions

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \cdot \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}, \tag{2.26}$$

see [1, Theorem 4.11.6]. It is clear from the explicit representation of the Gegenbauer polynomials in terms of Tchebycheff polynomials [1, equation (6.4.11)] that for $k \geq 0$, the expansion coefficients of P_n with respect to the exponentials $z \mapsto e^{mz}$, $m \in \mathbb{Z}$, are all nonnegative. A closer inspection shows that the same holds for the nonsymmetric E_n. This is in fact a special case of a deep result for general Heckman-Opdam polynomials due to Sahi [17]: if the multiplicity function k is nonnegative, then it follows from [17, Corollary 5.2 and Proposition 6.1] that the coefficients $a_{\lambda,\nu}$ of E_λ in (2.13) are all real and nonnegative. More precisely, if $\Pi_k := \mathbb{Z}_+[k_\alpha]$ denotes the set of polynomials in the parameters k_α

with nonnegative integral coefficients, then for suitable $d_\lambda \in \Pi_k$, all coefficients of $d_\lambda E_\lambda$ are contained in Π_k as well. This positivity result is the key for our subsequent proof of Theorem 2.1.

3 New proof of Theorem 2.1

In contrast to our approach in [15], we first derive a positive integral representation for the Dunkl kernel. As before, R and $k \geq 0$ are fixed.

Proposition 3.1. For each $x \in \mathfrak{a}$, there exists a unique probability measure $\mu_x \in M^1(\mathfrak{a})$ such that (2.6) holds. The support of μ_x is contained in $C(x)$. ☐

Proof. It suffices to prove the existence of the representing measures as stated; their uniqueness is immediate from the injectivity of the (usual) Fourier-Stieltjes transform on $M^1(\mathfrak{a})$. Let $\lambda \in P$. Then by Sahi's positivity result mentioned above,

$$G(\widetilde{\lambda}, \cdot) = \frac{1}{c_\lambda} E_\lambda = \sum_{\nu \trianglelefteq \lambda} b_{\lambda,\nu} e^{\nu}, \tag{3.1}$$

with coefficients $b_{\lambda,\nu}$ satisfying

$$0 \leq b_{\lambda,\nu} \leq 1, \quad \sum_{\nu \trianglelefteq \lambda} b_{\lambda,\nu} = 1. \tag{3.2}$$

Now fix $\lambda \in P$ and $z \in \mathfrak{a}_{\mathbb{C}}$. Then by Corollary 2.3,

$$\mathrm{Exp}_W(\lambda, z) = \lim_{n \to \infty} \frac{1}{c_{n\lambda}} E_{n\lambda}\left(\frac{z}{n}\right) = \lim_{n \to \infty} \sum_{\nu \trianglelefteq n\lambda} b_{n\lambda,\nu} e^{\langle \nu, z/n \rangle}. \tag{3.3}$$

Introducing the discrete probability measures

$$\mu_\lambda^n := \sum_{\nu \trianglelefteq n\lambda} b_{n\lambda,\nu} \delta_{\nu/n} \in M^1(\mathfrak{a}), \tag{3.4}$$

(where δ_x denotes the point measure in $x \in \mathfrak{a}$), we may write the above relation in the form

$$\mathrm{Exp}_W(\lambda, z) = \lim_{n \to \infty} \int_{\mathfrak{a}} e^{\langle \xi, z \rangle} d\mu_\lambda^n(\xi). \tag{3.5}$$

The following lemma shows that the support of μ_λ^n is contained in $C(\lambda)$.

Lemma 3.2. Let $\lambda, \nu \in P$ with $\nu \trianglelefteq \lambda$. Then $\nu \in C(\lambda)$. ☐

Proof. Let $C := \{x \in \mathfrak{a} : \langle \alpha, x\rangle \geq 0 \,\forall \alpha \in R_+\}$ be the closed Weyl chamber associated with R_+ and

$$C^* := \left\{y \in \mathfrak{a} : \langle y, x\rangle \geq 0 \,\forall x \in C\right\} \tag{3.6}$$

its closed dual cone. Notice that $Q_+ \subset C^*$. Therefore, $\nu \trianglelefteq \lambda$ implies that $\lambda_+ - \nu_+ \in C^*$. We employ the following characterization of $C(x)$ for $x \in C$ [11, Lemma IV.8.3]:

$$C(x) = \bigcup_{w \in W} w\big(C \cap (x - C^*)\big). \tag{3.7}$$

This shows that $\nu \in C(\lambda)$ if and only if $\nu_+ \in \lambda_+ - C^*$, which yields the statement. ■

We now continue with the proof of Proposition 3.1. Fix $\lambda \in P$. By the preceding result, we may consider the μ_λ^n as probability measures on the compact set $C(\lambda)$. According to Prohorov's theorem (see, e.g., [3]), the set $\{\mu_\lambda^n,\ n \in \mathbb{Z}_+\}$ is relatively compact. Passing to a subsequence if necessary, we may therefore assume that there exists a measure $\mu_\lambda \in M^1(\mathfrak{a})$ which is supported in $C(\lambda)$ and such that $\mu_\lambda^n \to \mu_\lambda$ weakly as $n \to \infty$. Thus in view of (3.5),

$$\mathrm{Exp}_W(\lambda, z) = \int_{\mathfrak{a}} e^{\langle \xi, z\rangle} d\mu_\lambda(\xi) \quad \forall z \in \mathfrak{a}_{\mathbb{C}}. \tag{3.8}$$

In order to extend this representation to arbitrary arguments $x \in \mathfrak{a}$ instead of $\lambda \in P$, observe first that for $r \in \mathbb{Q}$,

$$\mathrm{Exp}_W(r\lambda, z) = \mathrm{Exp}_W(\lambda, rz) = \int_{\mathfrak{a}} e^{r\langle \xi, z\rangle} d\mu_\lambda(\xi). \tag{3.9}$$

Defining $\mu_{r\lambda} \in M^1(\mathfrak{a})$ as the image measure of μ_λ under the dilation $\xi \mapsto r\xi$ on $\mathfrak{a}$, we therefore obtain (2.6) for all $x \in \mathbb{Q}.P = \{r\lambda : r \in \mathbb{Q},\ \lambda \in P\}$. The set $\mathbb{Q}.P$ is obviously dense in $\mathfrak{a}$. For arbitrary $x \in \mathfrak{a}$, choose an approximating sequence $\{x_n,\ n \in \mathbb{Z}_+\} \subset \mathbb{Q}.P$ with $\lim_{n\to\infty} x_n = x$. Using Prohorov's theorem once more, we obtain, after passing to a subsequence, that $\mu_{x_n} \to \mu_x$ weakly for some $\mu_x \in M^1(\mathfrak{a})$. The support of μ_x can be confined to an arbitrarily small neighbourhood of $C(x)$, and must therefore coincide with $C(x)$. We thus have

$$\mathrm{Exp}_W(x, z) = \lim_{n\to\infty} \mathrm{Exp}_W\left(x_n, z\right) = \int_{\mathfrak{a}} e^{\langle \xi, z\rangle} d\mu_x(\xi) \quad \forall z \in \mathfrak{a}_{\mathbb{C}}, \tag{3.10}$$

which finishes the proof of the proposition. ■

Proof of Theorem 2.1. By Proposition 3.1 and the definition of V,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}V_x\big(\langle x,z\rangle^n\big) &= V_x\left(e^{\langle x,z\rangle}\right) = \int_{\mathfrak{a}} e^{\langle \xi,z\rangle}\,d\mu_x(\xi)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathfrak{a}}\langle \xi,z\rangle^n\,d\mu_x(\xi) \quad (z\in\mathfrak{a}_{\mathbb{C}});\end{aligned} \tag{3.11}$$

here the subscript x means that V is taken with respect to x. Comparison of the homogeneous parts in z of degree n yields that

$$V_x\big(\langle x,z\rangle^n\big) = \int_{\mathfrak{a}}\langle \xi,z\rangle^n\,d\mu_x(\xi) \quad \forall n\in\mathbb{Z}_+. \tag{3.12}$$

As the $\mathbb{C}$-span of $\{x\mapsto\langle x,z\rangle^n,\ z\in\mathfrak{a}_{\mathbb{C}}\}$ is $\mathcal{P}_n$, it follows by linearity that

$$Vp(x) = \int_{\mathfrak{a}} p(\xi)\,d\mu_x(\xi) \quad \forall p\in\mathcal{P},\ x\in\mathfrak{a}. \tag{3.13}$$

Finally, as $\mathcal{P}$ is dense in each $(A_r, \|\cdot\|_{A_r})$ and $\|\cdot\|_{\infty,K_r}\le\|\cdot\|_{A_r}$, an easy approximation argument implies that this integral representation remains valid for all $f\in A_r$, with $r\ge|x|$. This finishes the proof. ■

We conclude this paper with a remark concerning positive product formulas. It is conjectured that (again in case $k\ge 0$) the multivariable hypergeometric function F has a positive product formula. More precisely, we conjecture that for all $x,y\in\mathfrak{a}$, there exists a probability measure $\sigma_{x,y}\in M^1(\mathfrak{a})$ whose support is contained in the ball $K_{|x|+|y|}(0)$ and which satisfies

$$F(\lambda,x)F(\lambda,y) = \int_{\mathfrak{a}} F(\lambda,\xi)\,d\sigma_{x,y}(\xi) \quad \forall\lambda\in\mathfrak{a}_{\mathbb{C}}. \tag{3.14}$$

In the rank-one case, that is, for Jacobi functions, this is well known and goes back to [9]. Equation (3.14) would immediately imply a positive product formula for the generalized Bessel function J_W (associated with the same multiplicity k). In fact, suppose there exist measures $\sigma_{x,y}$ as conjectured above, and denote for $r>0$ the image measure of $\sigma_{x,y}$ under the dilation $\xi\mapsto r\xi$ on $\mathfrak{a}$ by $\sigma_{x,y}^r$. Then by relation (2.21),

$$\begin{aligned}J_W(\lambda,x)J_W(\lambda,y) &= \lim_{n\to\infty} F\left(n\lambda+\rho,\frac{x}{n}\right)F\left(n\lambda+\rho,\frac{y}{n}\right)\\ &= \lim_{n\to\infty}\int_{\mathfrak{a}} F\left(n\lambda+\rho,\frac{\xi}{n}\right)d\sigma_{x/n,y/n}^n(\xi)\end{aligned} \tag{3.15}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}$. As supp $\sigma^n_{x/n,y/n} \subseteq K_{|x|+|y|}(0)$ for all $n \in \mathbb{N}$, we may assume that there exists a probability measure $\tau_{x,y} \in M^1(\mathfrak{a})$ with supp $\tau_{x,y} \subseteq K_{|x|+|y|}(0)$ such that $\sigma^n_{x/n,y/n} \to \tau_{x,y}$ weakly as $n \to \infty$. As further $\lim_{n\to\infty} F(n\lambda + \rho, \xi/n) = J_W(\lambda, \xi)$ locally uniformly with respect to ξ, (3.15) implies the product formula

$$J_W(\lambda, x)J_W(\lambda, y) = \int_{\mathfrak{a}} J_W(\lambda, \xi)\,d\tau_{x,y}(\xi) \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}. \tag{3.16}$$

The uniqueness of $\tau_{x,y}$ is immediate from the injectivity of the Dunkl transform on $M^1(\mathfrak{a})$ (cf. [16, Theorem 2.6]).

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